SINGLE DOF SYSTEM

This section will introduce the basics of Dynamic Analysis by considering a Single Degree of Freedom (SDOF) problem

Initially a free vibration model is used to describe the natural frequency

Damping is then introduced and the concept of critical damping and the undamped solution is shown

Finally a Forcing function is applied and the response of the SDOF is explored in terms of time dependency and frequency dependency and compared to the terms found in the equations of motion

SINGLE DOF SYSTEM (CONT.)

- Consider the System Shown

m = mass (inertia)

b = damping (energy dissipation)

k = stiffness (restoring force)

p = applied force

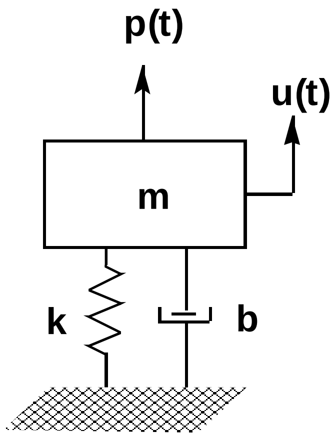
u = displacement of mass

$\dot{u}$ = velocity of mass

$\ddot{u}$ = acceleration of mass

u, $\dot{u}$, $\ddot{u}$ and p are time varying in general.

m, b, and k are constants.



SINGLE DOF SYSTEM (CONT.)

Some Theory:

- The equation of motion is:

$$m\ddot{u}(t) + b\dot{u}(t) + ku(t) = p(t)$$

- In **undamped, free vibration** analysis, the SDOF equation of motion reduces to:

$$m\ddot{u}(t) + ku(t) = 0$$

  - Has a solution of the form:

$$u(t) = A\sin\omega_n t + B\cos\omega_n t$$

  - This form defines the response as being HARMONIC, with a resonant frequency of:

$$\omega_n$$

UNDAMPED FREE VIBRATION SDOF SYSTEM

For an SDOF system the resonant, or natural frequency, is given by:

$$\omega_n = \sqrt{\frac{k}{m}}$$

Solve for the constants:

$$\text{When} \quad t = 0, \quad \sin(\omega_n t) = 0 \ \text{thus} \ B = u(t = 0)$$

$$\text{Differentiating solution :}$$

$$\dot{u}(t) = A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t$$

$$\text{When} \quad t = 0, \ B\,\omega_n \sin(\omega_n t) = 0 \ \text{thus}$$

$$A = \frac{\dot{u}(t = 0)}{\omega_n}$$

$$u(t) = \frac{\dot{u}(0)}{\omega_n} \sin\omega_n t + u(0)\cos\omega_n t$$

UNDAMPED FREE VIBRATION SDOF SYSTEM (CONT.)

The response of the Spring will be harmonic, but the actual form of the response through time will be affected by the initial conditions:

If $u(0) = 0 \text{ and } \dot{u}(0) = 0$ there is no response

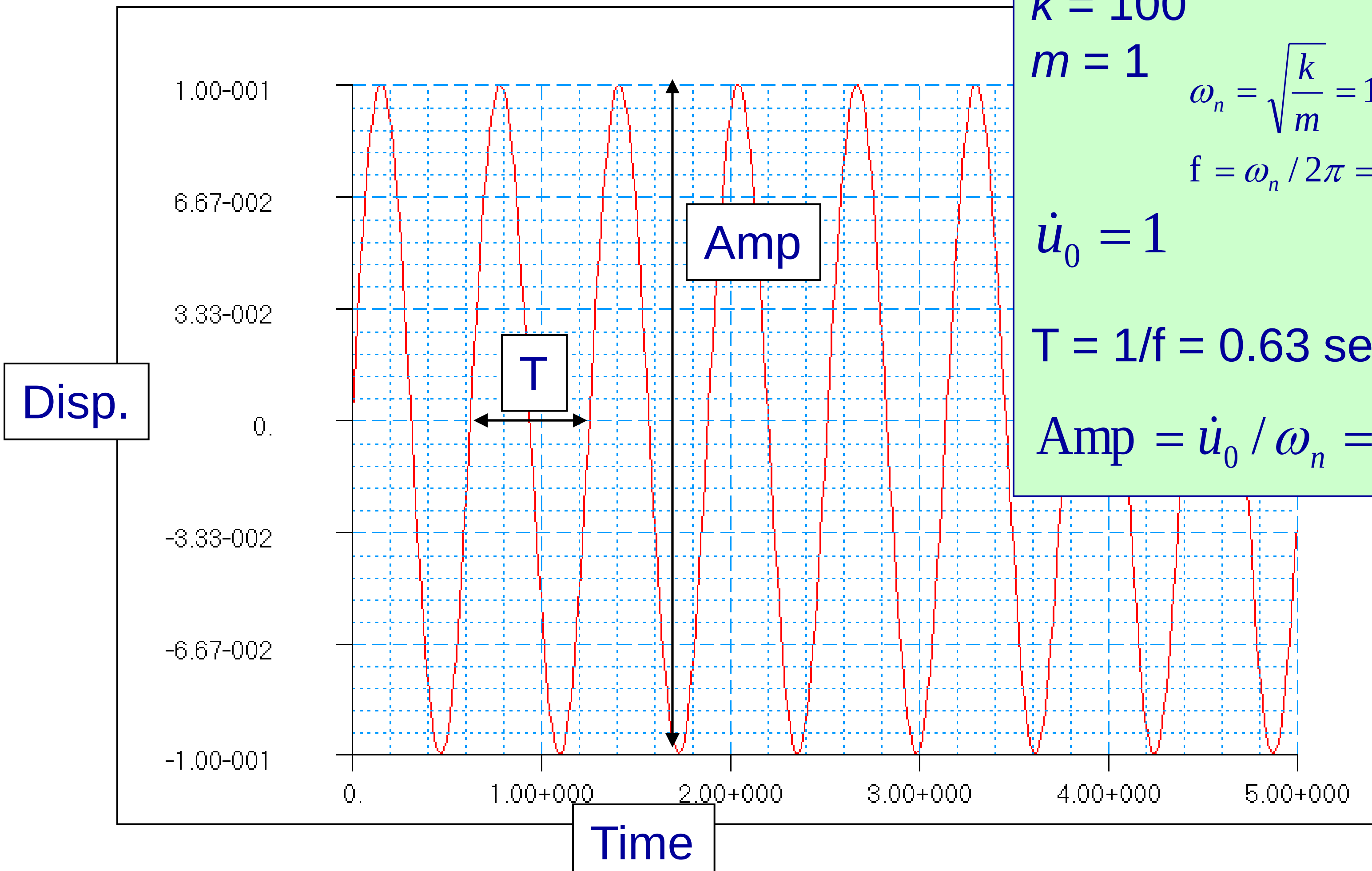
If $u(0) = 0 \text{ and } \dot{u}(0) \neq 0$ response is a sine function magnitude $\dot{u}_0 / \omega_n$

If $u(0) \neq 0 \text{ and } \dot{u}(0) = 0$ response is a cosine function (180 phase change), magnitude u_0

If $u(0) \neq 0 \text{ and } \dot{u}(0) \neq 0$ response is phase and magnitude dependent on the initial values

SINGLE DOF SYSTEM – UNDAMPED FREE VIBRATIONS

- The graph is from a transient analysis of a spring mass system with Initial velocity conditions only



$k = 100$

$m = 1$

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$f = \omega_n / 2\pi = 1.59 \text{ Hz}$$

$$\dot{u}_0 = 1$$

T = 1/f = 0.63 secs

$$\text{Amp} = \dot{u}_0 / \omega_n = 0.1$$

DAMPED FREE VIBRATION SDOF

If viscous damping is assumed, the equation of motion becomes:

$$m\ddot{u}(t) + b\dot{u}(t) + ku(t) = 0$$

There are 3 types of solution to this, defined as:

- Critically Damped
- Overdamped
- Underdamped

A swing door with a dashpot closing mechanism is a good analogy

- If the door oscillates through the closed position it is **underdamped**
- If it creeps slowly to the closed position it is **overdamped**.
- If it closes in the minimum possible time, with no overswing, it is **critically damped**.

DAMPED FREE VIBRATION SDOF (CONT.)

For the **critically damped** case, there is no oscillation, just a decay from the initial conditions:

$$u(t) = (A + Bt)e^{-bt/2m}$$

The damping in this case is defined as:

$$b = b_{cr} = 2\sqrt{km} = 2m\omega_n$$

A system is **overdamped** when $b > b_{cr}$

Generally only the final case is of interest - **underdamped**

DAMPED FREE VIBRATION SDOF (CONT.)

For the underdamped case b < b_{cr} and the solution is the form:

$$u(t) = e^{-bt/2m}(A\sin\omega_d t + B\cos\omega_d t)$$

- ω_d represents the Damped natural frequency of the system

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- ζ is called the (Critical) damping ratio and is defined by:

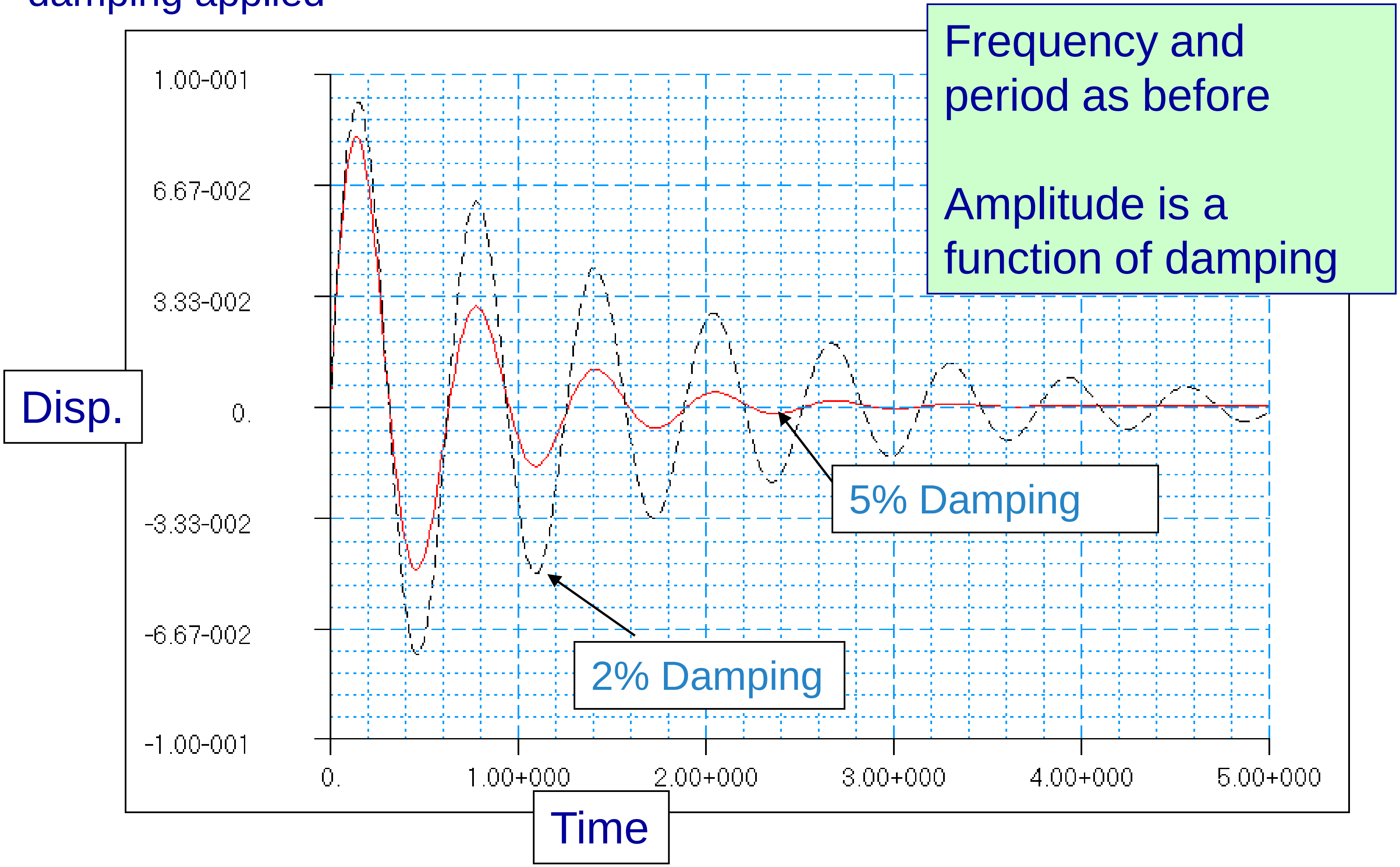
$$\zeta = \frac{b}{b_{cr}}$$

- In most analyses ζ is less than .1 (10%) so

$$\omega_d \approx \omega_n$$

DAMPED FREE VIBRATION SDOF (Cont.)

- The graph is from a transient analysis of the previous spring mass system with damping applied



DAMPING WITH FORCED VIBRATION

- Apply a harmonic forcing function: $p \sin \omega t$
  - note that ω is the DRIVING or INPUT frequency

- The equation of motion becomes

$$m\ddot{u}(t) + b\dot{u}(t) + ku(t) = p \sin \omega t$$

- The solution consists of two terms:
  - The initial response, due to initial conditions which decays rapidly in the presence of damping
  - The steady-state response as shown:

$$u(t) = p/k \frac{\sin(\omega t + \theta)}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta \omega / \omega_n)^2}}$$

  - This equation is described on the next page

DAMPING WITH FORCED VIBRATION (Cont.)

- This equation deserves inspection as it shows several important dynamic characteristics:

$$u(t) = p/k \frac{\sin(\omega t + \theta)}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$

Phase lead of the response relative to the input (see next page)

This is the static loading and dominates as ω tends to 0.0

At $\omega >> \omega_n$ both terms drive the response to 0.0

At $\omega = \omega_n$ this term = 0.0
With no damping present this results in an infinite response

At $\omega = \omega_n$ this term = (2ζ)^2 and controls the scaling of the response

From this is derived the Dynamic Magnification Factor 1/2 ζ

DAMPING WITH FORCED VIBRATION (Cont.)

- θ is defined as a phase lead in Nastran :

$$\theta = -\tan^{-1} \frac{2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2}$$

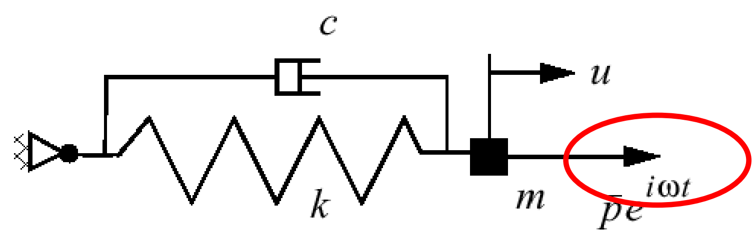
DAMPING WITH FORCED VIBRATION (Cont.)

- Summary:
  - For $\frac{\omega}{\omega_n} << 1$
    - Magnification factor $\longrightarrow$ 1 (static solution)
    - Phase angle $\longrightarrow$ 360^o (response is in phase with the force)
  - For $\frac{\omega}{\omega_n} >> 1$
    - Magnification factor $\longrightarrow$ 0 (no response)
    - Phase angle $\longrightarrow$ 180^o (response has opposite sign of force)
  - For $\frac{\omega}{\omega_n} \approx 1$
    - Magnification factor $\longrightarrow$ $1/2\xi$
    - Phase angle $\longrightarrow$ 270^o

HARMONIC OSCILLATIONS

When the Damped system is loaded with an exponential function of a single frequency, the resultant oscillations are called harmonic:

$$m\ddot{u} + c\dot{u} + ku = \bar{p}e^{i\omega t}$$



then define:

$$\omega_0 = \sqrt{\frac{k}{m}} \; ; \; \beta = \frac{\omega}{\omega_0}$$

static solution is:

$$u_s = \frac{p}{k}$$

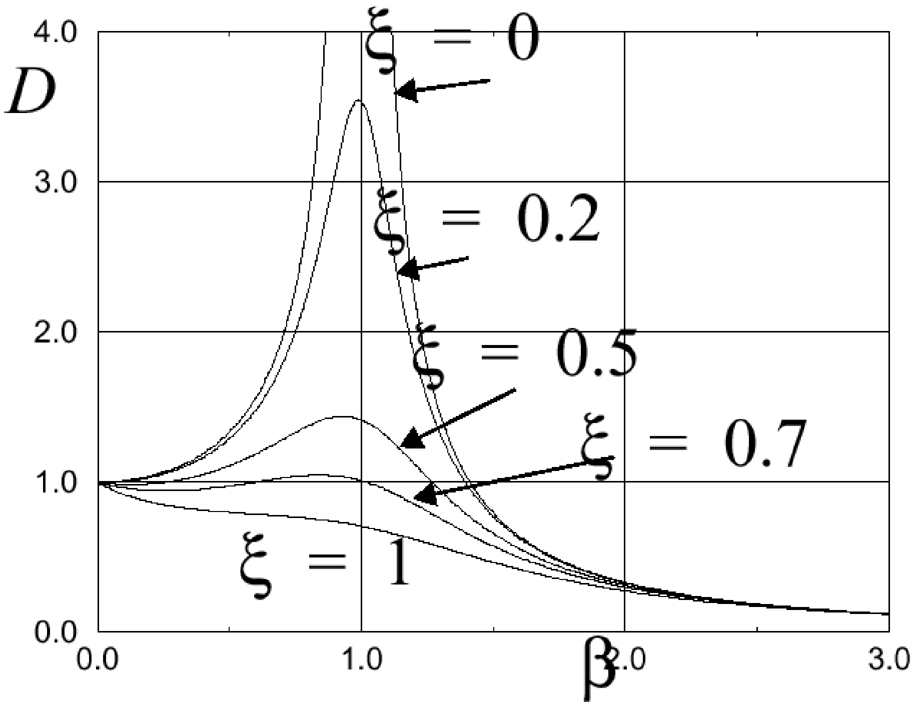
Dynamic magnification factor:

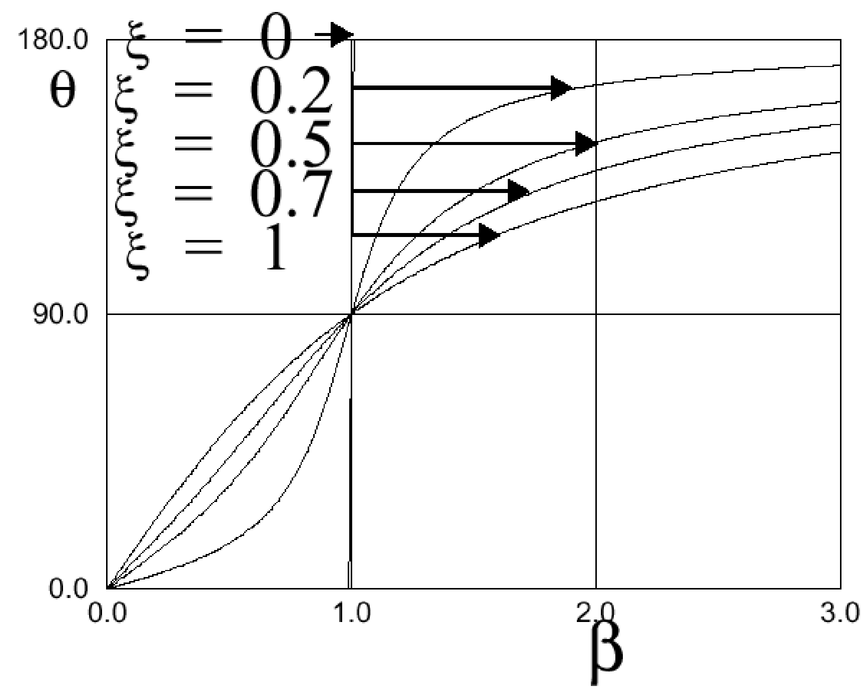
$$D = \frac{u}{u_s} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\xi)^2}}$$

with a phase angle of:

$$\theta = \operatorname{atan}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

HARMONIC OSCILLATIONS (CONT.)





Dynamic magnification factor:

$$D = \frac{u}{u_s} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\xi)^2}}$$

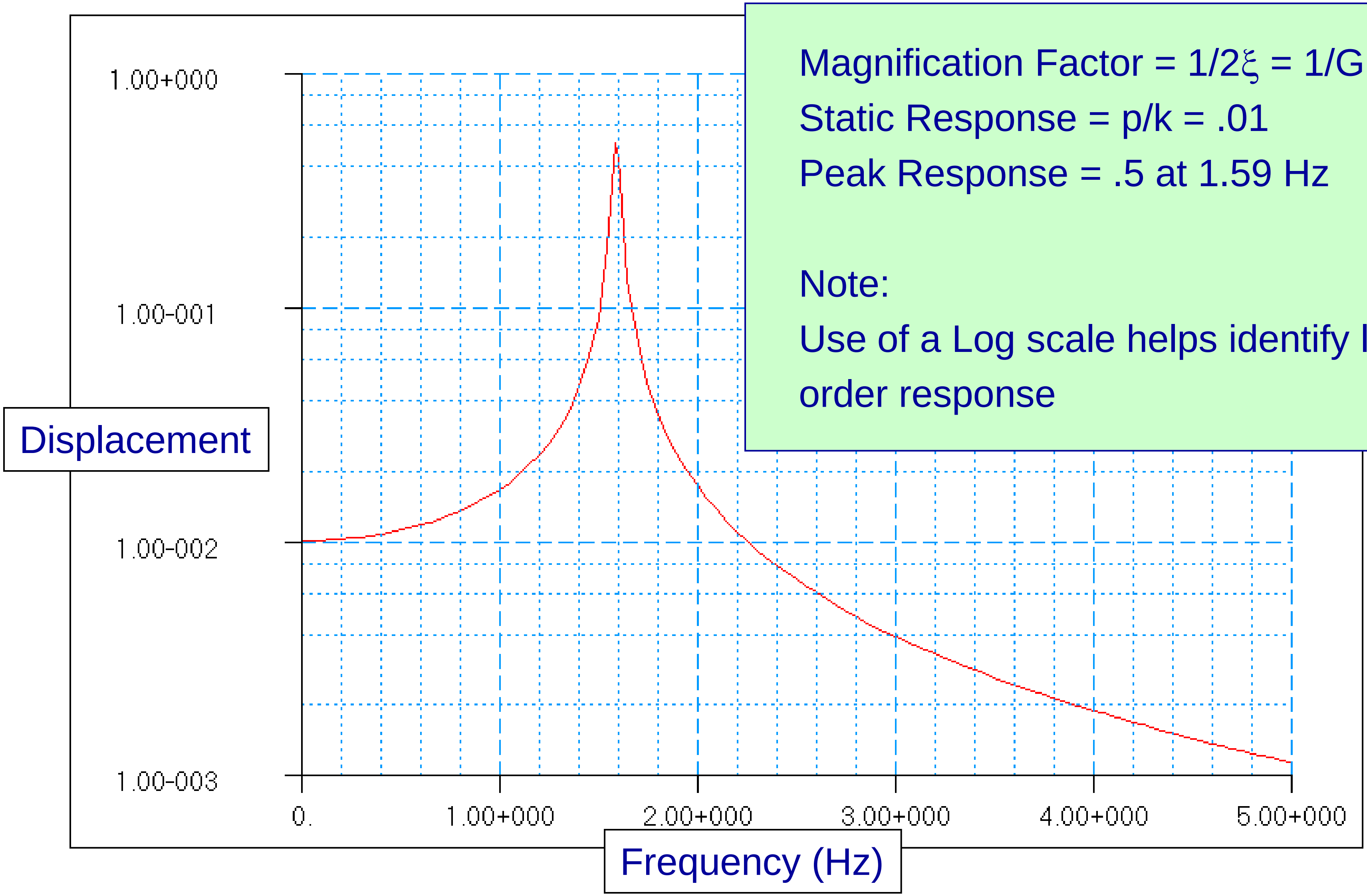
with a phase angle of:

$$\theta = \operatorname{atan}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

DAMPING WITH FORCED VIBRATION (Cont.)

- A Frequency Response Analysis can be used to explore the response of our spring mass system to the forcing function.
- This method allows us to compare the response of the spring with the input force applied to the spring over a wide range of input frequencies
- It is more convenient in this case than running multiple Transient Analyses, each with different input frequencies
- Apply the input load as 1 unit of force over a frequency range from .1 Hz to 5 Hz
- Damping is 1% of Critical

DAMPING WITH FORCED VIBRATION (Cont.)



Magnification Factor = 1/2ξ = 1/G = 50

Static Response = p/k = .01

Peak Response = .5 at 1.59 Hz

Note:

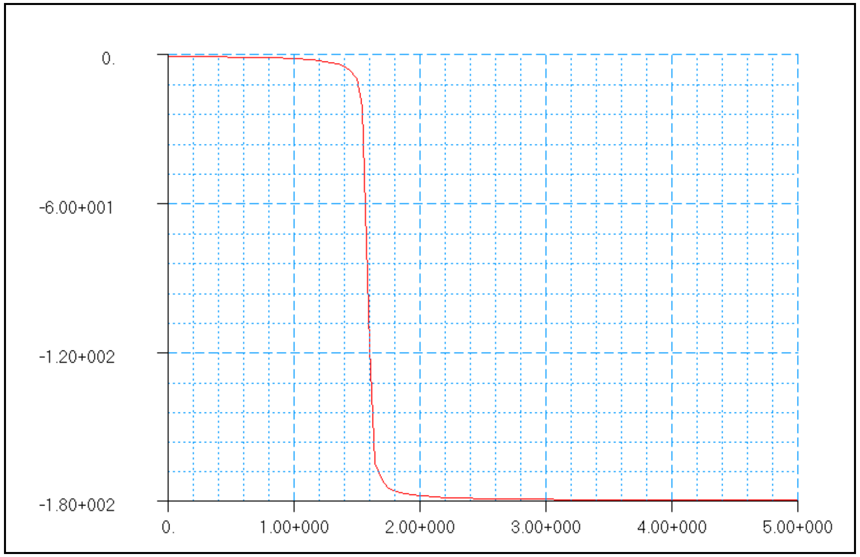
Use of a Log scale helps identify low order response

DAMPING WITH FORCED VIBRATION (Cont.)

- There are many important factors in setting up a Frequency Response Analysis that will be covered in a later section

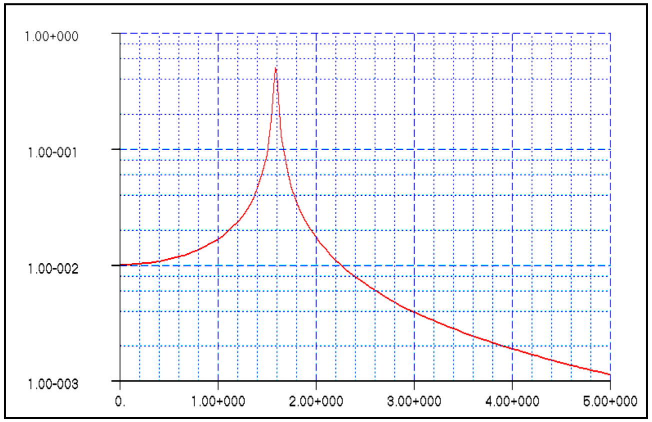
- For now, note the response is as predicted by the equation of motion
  - At 0 Hz result is p/k
  - At 1.59 Hz result is p/k factored by Dynamic Magnification
  - At 5 Hz result is low and becoming insignificant

- The Phase change is shown here:
  - In phase up to 1.59 Hz
  - Out of phase180 Degrees after 1 .59 Hz



DAMPING WITH FORCED VIBRATION (Cont.)

- Try a Transient analysis with a unit force applied to the spring at 1.59 Hz
- Again damping of 1% Critical is applied
- The result is shown on the next page:
  - The response takes around 32 seconds to reach a steady-state solution
  - After this time the displacement response magnitude stays constant at .45 units
  - The theoretical value of .5 is not reached due to numerical inaccuracy (see later) and the difficulty of hitting the sharp peak



DAMPING WITH FORCED VIBRATION (Cont.)

- Transient analysis with a unit force applied to the spring at 1.59 Hz